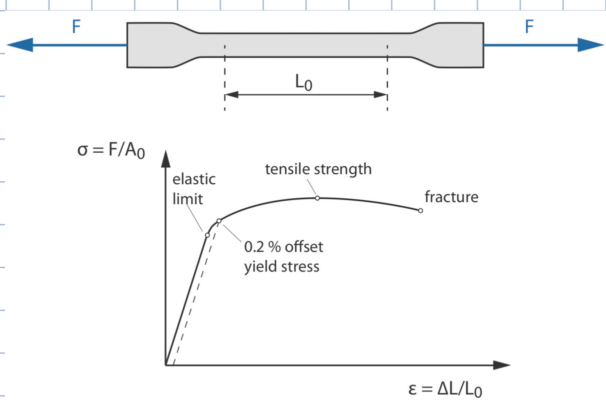
Governing equations of elasticity :

- equilibrium equations : stress transformation
- compatibility equations : strain transformation
- constitutive equations : material model

Majority of engineering materials :

- homogenous : properties are the same in all locations
- isotropic : properties are the same in all directions
- linear : linear relationship between stress & strain
- elastic : no energy is lost in deformation



Cauchy Strain & Stress Tensors :

$$\bar{\sigma} : \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \qquad \bar{\varepsilon} : \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}$$

note : $\varepsilon_{xy} = \dfrac{\gamma_{xy}}{2}$

- Linear material described by 6x6 matrix S with 36 material constants : S_{ij}

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

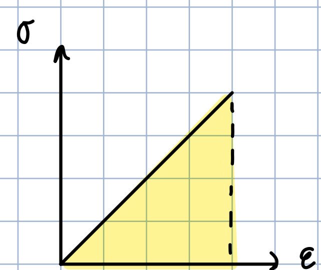
We want to reduce number of these constants.

Strain Energy: (Hooke's Law)

1. Apply stress σ_{xx} with resulting strain energy :

$$U_A = \frac{1}{2}\sigma_{xx}\varepsilon_{xx} = \frac{1}{2} S_{11}\sigma_{xx}^2$$

strain energy



strain energy = area under graph

2. Applying stress σ_{yy} while maintaining stress σ_{xx} :

$$U_b = \frac{1}{2} S_{22} \sigma_{yy}^2 + \underbrace{\sigma_{xx} S_{12} \sigma_{yy}}_{\varepsilon_{xx} \text{ strain due to } \sigma_{yy} \text{ (Poisson)}}$$

units energy per unit volume

Total elastic strain energy :

$$U_A + U_B = \frac{1}{2} S_{11} \sigma_{xx}^2 + \frac{1}{2} S_{22} \sigma_{yy}^2 + \sigma_{xx} S_{12} \sigma_{yy} = U$$

Repeat process applying σ_{yy} first then σ_{xx} :

$$U = \frac{1}{2} S_{22} \sigma_{yy}^2 + \frac{1}{2} S_{11} \sigma_{xx}^2 + \sigma_{xx} S_{12} \sigma_{yy}$$

→ same result, ∴ strain energy independent of order of application

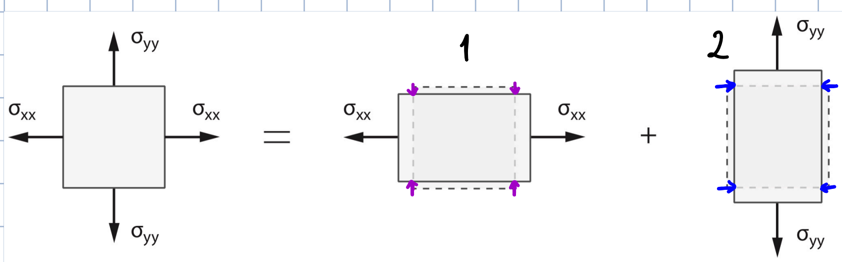
∴ $S_{ij} = S_{ji}$ → 21 elastic constants

Using mathematical approach to isotropic materials (too complex) :

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}$$

2 constants : S_{11} & S_{12}

Consider Bi-Axial Stress State :



Direct strains :

1. $\varepsilon_{xx} = \dfrac{\sigma_{xx}}{E}$ & strain in y due to Poisson : $\varepsilon_{yy} = -\nu \varepsilon_{xx} = -\nu \dfrac{\sigma_{xx}}{E}$

2. $\varepsilon_{yy} = \dfrac{\sigma_{yy}}{E}$ & strain in x due to Poisson : $\varepsilon_{xx} = -\nu \varepsilon_{yy} = -\nu \dfrac{\sigma_{yy}}{E}$

Superimposing two uni-axial strains:

$$\boxed{\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right)} \qquad \boxed{\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right)}$$

→ $\varepsilon_{xx} \neq -\nu\varepsilon_{yy}$ for bi-axial stress state

only applies to uni-axial

Shear Strains:

$$\boxed{\gamma_{xy} = \frac{\tau_{xy}}{G}}$$

→ combining into matrix formulation:

no link between applied shear stress and direct strain

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

no link between applied direct stress and shear strain

for isotropic materials, direct and shear strains decoupled

∴ principal directions for stress & strain coincide

Shear Modulus:

- for isotropic materials shear modulus G is <u>not</u> independent.

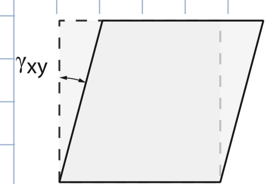
→ can see this if we consider that pure shear = bi-axial stress at 45°.

Shear strain due to pure shear:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

transforming to $x'y'$ coordinate system (+45°)

$$\varepsilon_{x'x'} = \sin\theta\cos\theta\,\gamma_{xy} = \frac{\tau_{xy}}{2G} \quad (1)$$



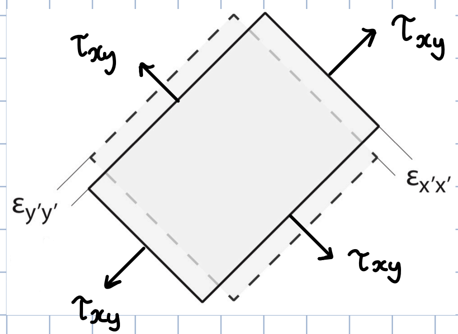
σ_{yy} & $\sigma_{xx} = 0$ os pure shear

Shear strain due to equivalent **direct stresses** :

$$\varepsilon_{x'x'} = \frac{1}{E}\left(\sigma_{x'x'} - \nu\sigma_{y'y'}\right)$$

$\sigma_{x'x'} = \sigma_{y'y'} = \tau_{xy}$ at $45°$

$$= \frac{\tau_{xy}}{E}(1-\nu) \quad ②$$



equating both strains ① & ②

$$\boxed{G = \frac{E}{2(1+\nu)}}$$

for an **isotropic, linear-elastic material** under plane stress

Compliance matrix :

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

inverse

Constitutative equations

Stiffness matrix :

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Plane Stress vs. Plane Strain :

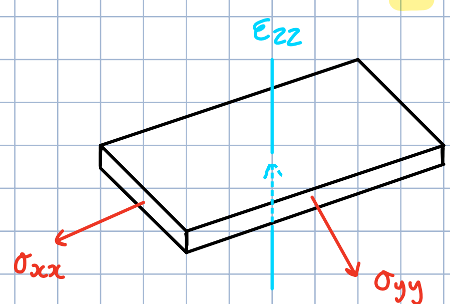
- expressions for through-thickness stress & strain (σ_{zz} & ε_{zz}) :

Plane Stress : $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$

→ Through-thickness strain $\neq 0$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right)$$

$$\boxed{\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \neq 0}$$



stresses in y & x direction result in out of plane z strains due to Poisson

Plane Strain :

$\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right) = 0$$

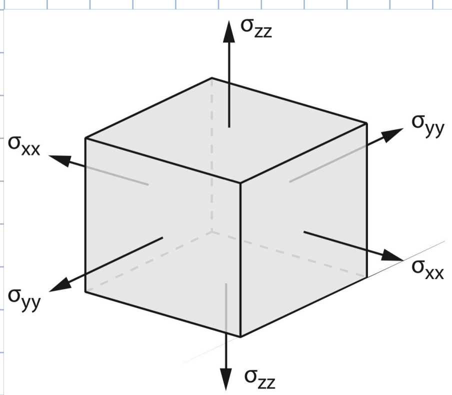
$\rightarrow$ $\boxed{\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})}$ $\rightarrow$ shows to ensure case of plane strain, you need through-thickness stress σ_{zz}.

Bulk Modulus :

- Only need two elastic constants to describe a material's properties :

  E & ν used in engineering

- by deriving bulk modulus, k, we can get further insight into material properties.



Volumetric strain describes change in material volume :

$$\frac{\Delta V}{V_0} = \frac{V - V_0}{V}$$

where $V_0 = dx\, dy\, dz$

Deformed Volume, V :

$$V = (1+\varepsilon_{xx})\,dx\,(1+\varepsilon_{yy})\,dy\,(1+\varepsilon_{zz})\,dz = (1+\varepsilon_{xx})(1+\varepsilon_{yy})(1+\varepsilon_{zz})\,dx\,dy\,dz$$

$$= (1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} + \text{higher order terms})\,dx\,dy\,dz$$

$$\therefore \boxed{\frac{\Delta V}{V_0} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})(1-2\nu)}{E}}$$

Special case where tri-axial stress is spherical stress : $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma$

$$\frac{\Delta V}{V} = \frac{3\sigma(1-2\nu)}{E}$$

Bulk modulus relates spherical stress to volumetric strain:

$$\sigma = K \frac{\Delta V}{V_0} \longrightarrow \boxed{K = \frac{E}{3(1-2\nu)}}$$

- Looking at bulk & shear moduli: (both stiffnesses)

$$K = \frac{E}{3(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)}$$

- for realistic materials, shear & bulk modulus must be positive & finite

→ bounds Poisson's Ratio values $\nu \in \langle -1, 0.5 \rangle$

↳ most engineering materials $\nu \in \langle 0.2, 0.5 \rangle$

Poisson 'Extremes':

- $\nu = 0.5$ → infinite bulk modulus → incompressible e.g rubber
- $\nu = 0$ e.g cork → doesn't expand in normal direction to applied stresses
- $\nu < 0$ → auxetics